The Scots College

HSC Mathematics Extension 2

Trial Examination

12th August 2011

Name:______________________

General Instructions

- Working time : 3 hours + 5 minutes reading time.
- Write using blue or black pen
- Board approved calculators may be used (Non Graphic)
- All necessary working should be shown in every question
- Standard Integrals Table attached
- Answer each question on a SEPARATE answer booklet

TOTAL MARKS: 120

Attempt Questions 1 - 8
All questions are of equal value

WEIGHTING: 40 **%**

Question 1 (Marks 15) Use a SEPARATE writing booklet.

a) Evaluate [2]

$$\int_0^{\frac{\pi}{4}} \frac{\sin x - \cos x}{\sin x + \cos x}\, dx$$

b) Find [3]

$$\int \frac{dx}{\sqrt{x\,(2-x)}}$$

using the substitution $\sqrt{\frac{x}{2}} = \sin\theta$

c) Show that [4]

$$\int_1^3 \frac{2x^2-3x+11}{(x+1)(x^2-2x+5)}\, dx = 2\,ln2 + \frac{\pi}{8}$$

d) Find [3]

$$\int x\ln(x+1)\, dx$$

e) Find [3]

$$\int \frac{dx}{1+\text{sinx}+\text{cosx}}$$

Question 2 (Marks 15) Use a SEPARATE writing booklet.

a) i) Find the square root of $-5 - 12i$. [2]

ii) Hence solve $z^2 - iz + 1 + 3i = 0$, expressing your answer in the form $a + ib$, where *a* and *b* are real numbers. [2]

b) i) Write $\frac{\sqrt{3} + i}{\sqrt{3} - i}$ in the modulus – argument form. [2]

ii) Hence express $\left(\frac{\sqrt{3} + i}{\sqrt{3} - i}\right)^{10}$ in the form $x + iy$, where *x* and *y* are both real. [2]

c) i) Sketch on the Argand diagram the locus of the complex number *z*, which satisfies the condition [2]

$$\arg\left(\frac{z-2}{z-2i}\right) = \frac{\pi}{2}$$

(ii) Hence, or otherwise, find the complex number *z* (in the form *a* + *ib*, where *a* and *b* are both real) which has the maximum value of $|z|$. [1]

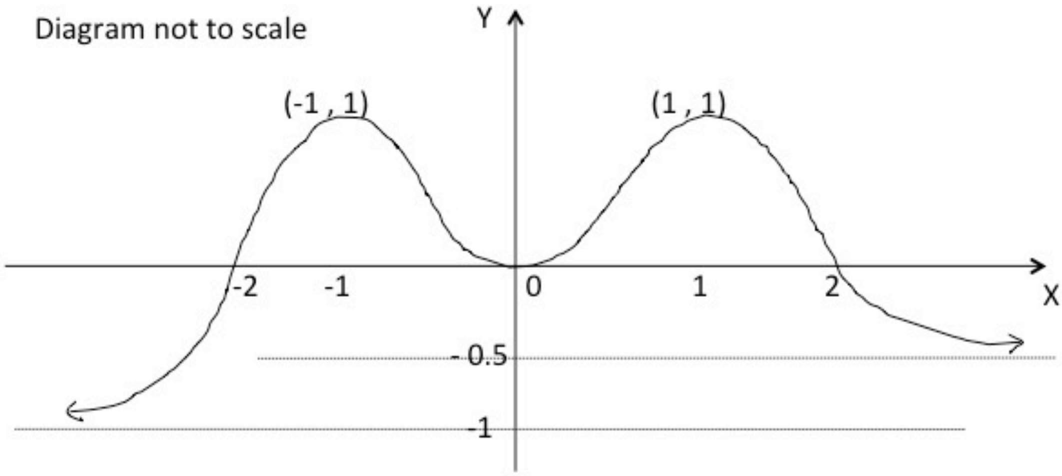
d) Let $z = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$

i) Show that $1 - z + z^2 - z^3 + z^4 = 0$ [2]

ii) Show that $(1 - z)(1 + z^2)(1 - z^3)(1 + z^4) = 1$ [2]

Question 3 (Marks 15) Use a SEPARATE writing booklet.

a) The graph of $y = f(x)$ is given below. [10]



Using the graph of $y = f(x)$, sketch on separate axes, the graphs of

i. $|y| = |f(x)|$

ii. $y^2 = f(x)$

iii. $y = \frac{1}{f(x)}$

iv. $y = e^{f(x)}$

v. $y = sin^{-1} f(x)$

b) (i) Show that if $I_n = \int_0^1 x^n \sqrt{1 - x^2}\, dx$, then [3]

$$I_n = \frac{n-1}{n+2} I_{n-2}$$

(ii) Hence evaluate $\int_0^1 x^4 \sqrt{1 - x^2}\, dx$ [2]

Question 4 (Marks 15) Use a SEPARATE writing booklet.

a) A hyperbola has the asymptotes $y = x$ and $y = -x$, and it passes through the point (5,4). Find [7]

i. the equation of the hyperbola

ii. its eccentricity

iii. the length of the principal axis

iv. the coordinates of the foci

v. equation of the directrices

vi. the length of the latus rectum

b) The point $P\ \left(cp, \frac{c}{p}\right)$, lies on the hyperbola $xy = c^2$. The tangent at P meets the x – axis at A and the y – axis at B . The normal to the hyperbola at P meets the line $y = x$ at the point C . [8]

i. Show that the equation of the tangent at P is $x + p^2y = 2cp$.

ii. Find the coordinates of A and B .

iii. Find the equation of the normal at P .

iv. Show that the x – coordinate of the point C is given by $x = \frac{c}{p}\ (p^2 + 1)$.

v. Prove that $\triangle\ ABC$ is an isosceles triangle.

Question 5 (Marks 15) Use a SEPARATE writing booklet.

a) $P\ (a\cos\theta,\ b\sin\theta\ )$ is a variable point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

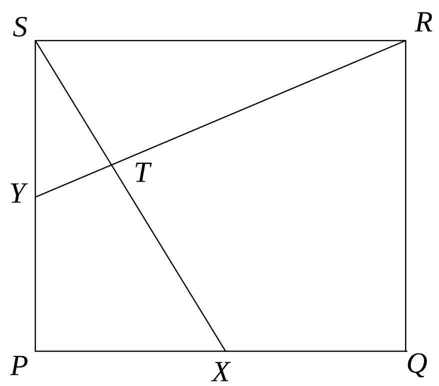
i. Show that the equation of the tangent to the ellipse at P is $bx\cos\theta + ay\sin\theta = ab$. [2]

ii. Deduce the equation of the normal to the ellipse at P . [2]

iii. Find the coordinates of X and Y, the points where the tangent and the normal respectively, meet the y – axis. [2]

iv. Show that the circle with XY as the diameter, passes through the foci of the ellipse.. [3]

b) *PQRS* is a square. *X* and *Y* are mid-points of *PQ* and *PS* respectively. *SX* and *RY* intersect at the point *T*. [6]



i. Prove that *QRTX* is a cyclic quadrilateral.

ii. Hence prove that $QT = QR$

Question 6 (Marks 15) Use a SEPARATE writing booklet.

a) A solid is formed by rotating the circle $x^2 - 2ax + y^2 = 0$ about the line $x = 3a$. [5]

Find the volume of the solid generated by taking slices perpendicular to the axis of rotation.

b) The base of a certain solid is the region between the curve $y = \frac{x^3}{4}$, $0 \le x \le 2$, and the line $y = x$. [6]

Each plane section of the solid perpendicular to the x-axis is a parabola whose chord lies on the base of the solid, with one end point *A* on the line $y = x$ and the other end point *B* on the curve $y = \frac{x^3}{4}$. The axis of the parabola is vertical and passing through the mid-point of *AB* and the maximum height of the parabola, from the base, is equal to the length of the chord *AB* .

By first finding the area of a slice taken perpendicular to the *x* – axis, find the volume of the solid.

c) A sequence $u_1, u_2, u_3, \ldots\ldots\ldots$ is defined by the relation [4]

$u_n = u_{n-1} + 6u_{n-2}$, for $n \ge 3$.

Given that $u_1 = 1$ and $u_2 = -12$, prove by using mathematical induction

$u_n = -6[(-2)^{n-2} + 3^{n-2}]$, for all positive integers *n* .

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Question 7 (Marks 15) Use a SEPARATE writing booklet.

a) i. By using De Moivre's Theorem or otherwise, prove that [3]

$$\tan 4\theta = \frac{4\tan\theta - 4\,tan^3\,\theta}{1-6\,tan^2\,\theta + tan^4\,\theta}.$$

ii. Using part (i) solve the equation [4]

$$x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$$

and hence find the value of

$$tan\frac{\pi}{16} \times tan\frac{3\pi}{16} \times tan\frac{5\pi}{16} \times tan\frac{7\pi}{16}$$

b) It is given that the product of two of the roots of the equation $x^4 + x^3 - 16x^2 - 4x + 48 = 0$, is equal to 6. [4]

Show that the equation can be written in the form $(x^2 + ax + b)(x^2 + cx + d) = 0$, where *a*, *b*, *c* and *d* are integers.

Hence or otherwise solve the equation.

c) i. Find all the values of *m* for which the polynomial $3x^4 - 4x^3 + m = 0$ has no real roots. [4]

ii. Determine the real roots of the polynomial when *m* = 1

Question 8 (Marks 15) Use a SEPARATE writing booklet.

a) A particle P of mass m kg is projected vertically upwards from the ground, with an initial velocity of u m/s, in a medium of resistance mkv^2, where k is a positive constant and v is the velocity of the particle. [8]

i. Show that the maximum height H, from the ground, attained by the particle P is given by $H = \frac{1}{2k}\ln\left(1 + \frac{ku^2}{g}\right)$, where g is the acceleration due to gravity.

ii. At the same time that P is projected upwards, another particle, Q, of equal mass, initially at rest, is allowed to fall downwards in the same medium, from a height of H metres from the ground, along the same vertical path as P. Show that at the time of collision of P and Q,

$$\frac{1}{v_2^2} - \frac{1}{v_1^2} = \frac{1}{V^2},$$

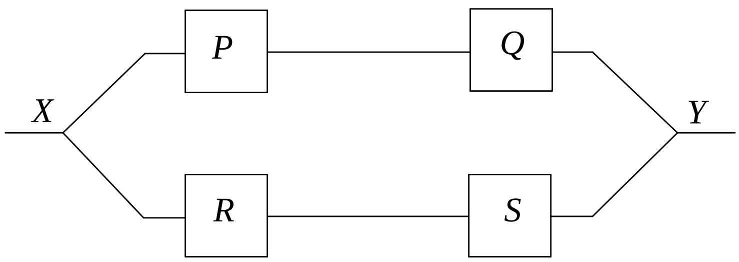
where v_1 and v_2 are the velocities of particles P and Q respectively, at the time of collision, and $V = \sqrt{\frac{g}{k}}$.

b) Find the stationary points, stating their nature, for the curve [3]

$$x^2 + y^2 = xy + 3$$

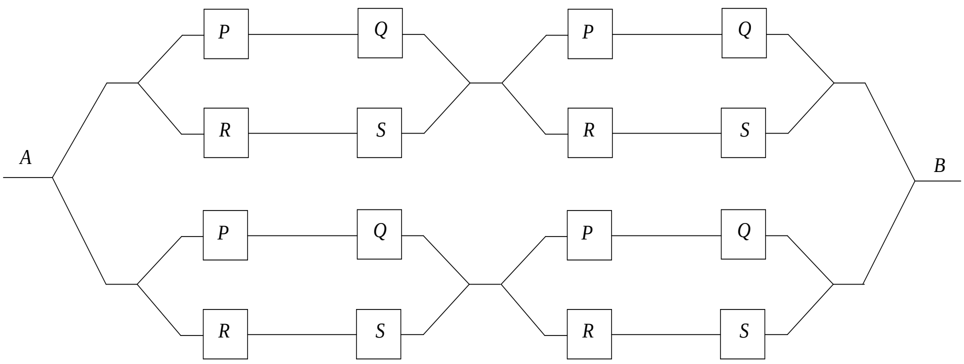
c) i. An electrical circuit has four bulbs P, Q, R and S placed as shown in the diagram. The probability of each bulb being defective, independently, is given by p. Current can flow from X to Y through either or both of the branches of the electrical circuit. However, no current will flow through a branch that has at least one defective bulb. [4]

Show that the probability that the current *does not* flow from X to Y is $(2p - p^2)^2$



Question 8 continued........

ii. In the household there are four such circuits in one connection. Find the probability that current does not flow from A to B .



End of Assessment

Standard Integrals

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

ANSWER SHEET

Name: SOLUTIONS

Teacher:

Mathematics Ext 2 . Trials 2011

Question No. 1

(a) $$\int_0^{\pi/4} \frac{\sin x - \cos x}{\sin x + \cos x}\, dx$$

$$= -\int_0^{\pi/4} \frac{\cos x - \sin x}{\sin x + \cos x}\, dx$$

$$= -\Big[\ln[\sin x + \cos x]\Big]_0^{\pi/4}$$

$$= -\Big[\ln\left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}\right) - \ln(0+1)\Big]$$

$$= -\ln \frac{2}{\sqrt{2}}$$

$$= -\ln\sqrt{2}$$

$$= -\tfrac{1}{2}\ln 2$$

~~(b)~~

ANSWER SHEET

Name: ______________________

Teacher: ______________________

Question No. 1.

$$\int \frac{dx}{\sqrt{x(2-x)}}$$

$$\sqrt{\frac{x}{2}} = \sin\theta$$

$$x = 2\sin^2\theta$$

$$dx = 4\sin\theta\cos\theta\,d\theta$$

$$= \int \frac{4\sin\theta\cos\theta\,d\theta}{\sqrt{2\sin^2\theta\cdot 2\cos^2\theta}}$$

$$= \int \frac{4\sin\theta\cos\theta}{2\sin\theta\cos\theta}\,d\theta$$

$$= \int 2\,d\theta$$

$$= 2\theta + C$$

$$= 2\sin^{-1}\sqrt{\frac{x}{2}} + C$$

(c) $$\int_1^3 \frac{2x^2-3x+11}{(x+1)(x^2-2x+5)}\,dx = 2\ln 2 + \frac{\pi}{8}$$

$$\frac{2x^2-3x+11}{(x+1)(x^2-2x+5)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-2x+5}$$

$$= \frac{A(x^2-2x+5)+(Bx+C)(x+1)}{(x+1)(x^2-2x+5)}$$

$$2x^2-3x+11 = A(x^2-2x+5)+(Bx+C)(x+1)$$

$x = -1$

$$2+3+11 = A(1+2+5)$$

$$16 = 8A \Rightarrow A = 2$$

$x = 0$

$$11 = 2(5) + C(1)$$

$$\therefore C = \cancel{4}\,1$$

Equating coeff of x^2

$$2 = 2 + B \quad \therefore B = 0$$

$$\therefore \int_1^3 \frac{2x^2-3x+11}{(x+1)(x^2-2x+5)}\,dx = \int_1^3 \left[\frac{2}{x+1} + \frac{\cancel{4}\,1}{(x-1)^2+4}\right]dx$$

$$= \Big[2\ln(x+1)\Big]_1^3 + \left[\frac{\cancel{4}}{2}\tan^{-1}\frac{x-1}{2}\right]_1^3$$

$$= (2\ln 4 - 2\ln 2) + \left[\cancel{2}\,\tfrac{1}{2}\tan^{-1}1 - 2\tan^{-1}0\right]$$

$$= 2\ln 2 + \cancel{2\tan^{-1}}\ \tfrac{1}{2}\cdot\tfrac{\pi}{4}$$

$$= 2\ln 2 + \frac{\pi}{8}.$$

ANSWER SHEET

Name: ____________________

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Question No. __________

(d) $\int x \ln(x+1)\,dx$

$v' = x \qquad u = \ln(x+1)$

$v = \frac{x^2}{2} \qquad u' = \frac{1}{x+1}$

$$= \frac{x^2}{2}\ln(x+1) - \int \frac{x^2}{2(x+1)}\,dx$$

$$= \frac{x^2}{2}\ln(x+1) - \frac{1}{2}\int \frac{x^2-1+1}{x+1}\,dx$$

$$= \frac{x^2}{2}\ln(x+1) - \frac{1}{2}\int \left(x-1+\frac{1}{x+1}\right)dx$$

$$= \frac{x^2}{2}\ln(x+1) - \frac{1}{2}\left[\left(\frac{x^2}{2}-x\right)+\ln(x+1)\right]+C$$

ANSWER SHEET

Name: ______________

Teacher: ______________

Question No. ______

(e) $\int \frac{dx}{1+\sin x+\cos x}$ let $t = \tan\frac{x}{2}$

$dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

$= \frac{1}{2}(1+t^2)\,dx$

$dx = \frac{2dt}{1+t^2}$

$$= \int \frac{2dt/(1+t^2)}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}$$

$$= \int \frac{2dt/(1+t^2)}{\frac{1+t^2+2t+1-t^2}{1+t^2}}$$

$$= \int \frac{2dt}{2+2t}$$

$$= \int \frac{dt}{1+t}$$

$$= \ln(1+t) + C$$

$$= \ln\left(1+\tan\frac{x}{2}\right) + C$$

ANSWER SHEET

Name: ______________

Teacher: ______________

Question No. 2

(a) (i) let $\sqrt{-5-12i} = x+iy$

$$x^2+2ixy-y^2 = -5-12i$$

$$x^2-y^2 = -5$$

$$2xy = -12 \Rightarrow xy = -6$$

$$y = -6/x$$

$$x^2 - \frac{36}{x^2} = -5$$

$$x^4+5x^2-36 = 0$$

$$(x^2+9)(x^2-4) = 0$$

$x^2 = -9$ or $x^2 = 4$

(rej) $\therefore x = 2,\ y = -3$

or $x = -2$ $y = 3$.

$\therefore$ Square roots of

$-5-12i$ are $\underline{2-3i}$ or $\underline{-2+3i}$.

ANSWER SHEET

Name: ____________________

Teacher: ____________________

Question No. 2

(a) (ii) $z^2 - iz + 1 + 3i = 0$

$$z = \frac{i \pm \sqrt{(-i)^2 - 4(1)(+1+3i)}}{2(1)}$$

$$= \frac{i \pm \sqrt{-1 - 4 - 12i}}{2}$$

$$= \frac{i \pm \sqrt{-5-12i}}{2}$$

$$= \frac{i + 2 - 3i}{2} \text{ or } \frac{i - 2 + 3i}{2}$$

$$= \frac{2-2i}{2} \text{ or } \frac{-2+4i}{2}$$

$$= \underline{1-i} \text{ or } \underline{-1+2i}$$

ANSWER SHEET

Name: ____________________

Teacher: ____________________

Question No. 2.

(b) (i) $\dfrac{\sqrt{3}+i}{\sqrt{3}-i}$

$$\sqrt{3}+i = 2\left(\cos \pi/6 + i \sin \pi/6\right)$$

$$\sqrt{3}-i = 2\left(\cos(-\pi/6) + i \sin(-\pi/6)\right)$$

$$\frac{\sqrt{3}+i}{\sqrt{3}-i} = \cos\left(\frac{\pi}{6}+\pi/6\right) + i \sin\left(\pi/6+\pi/6\right)$$

$$= \underline{\cos \pi/3 + i \sin \pi/3}$$

(ii)
$$\left(\frac{\sqrt{3}+i}{\sqrt{3}-i}\right)^{10} = \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)^{10}$$

$$= \left(\cos \frac{10\pi}{3} + i \sin \frac{10\pi}{3}\right)$$

$$= -\cos(\pi/3) - i \sin(\pi/3).$$

$$= \underline{\underline{-\frac{1}{2} - \frac{\sqrt{3}}{2} i}}$$

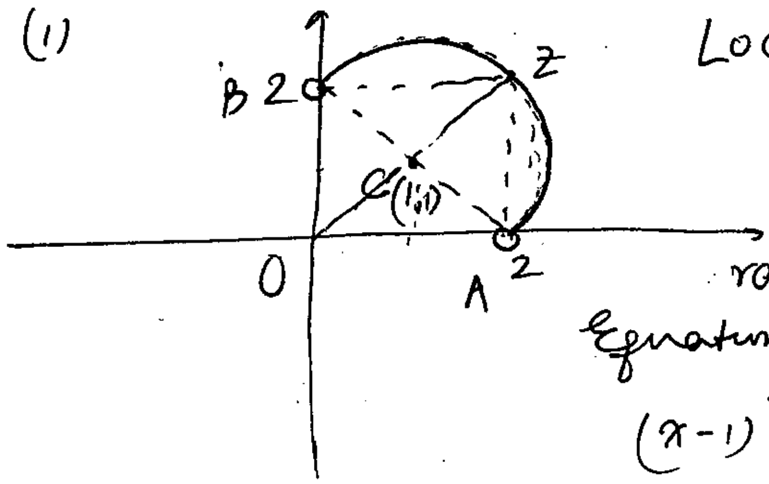
ANSWER SHEET

Name: ____________________

Teacher: ____________________

Question No. 2

(c) (i)



Locus is the semi-circle with AB as Diameter.

radius : $\frac{1}{2}\sqrt{(2^2+2^2)} = \frac{1}{2}\sqrt{8}$
$= \sqrt{2}$

Equation of circle.

$(x-1)^2 + (y-1)^2 = \cancel{\sqrt{}}2$

(ii) Max. value of z lies along the line $y = x$.

$\therefore |z| = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$

$z = 2 + 2i$

ANSWER SHEET

Name: ____________________

Teacher: ____________________

Question No. 2

(d) $z = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$.

$z^5 = \cos\frac{5\pi}{5} + i\sin\frac{5\pi}{5}$

$= -1 + 0$

$= -1$

(i) $1 - z + z^2 - z^3 + z^4$

$r = -z, \quad n = 5$

$S_5 = \dfrac{1-(-z)^5}{1-(-z)}$

$= \dfrac{1+z^5}{1+z} \quad (z \neq -1)$

$= \dfrac{0}{1+z} \quad (\because z^5 = -1)$

(ii) $(1-z)(1+z^2)(1-z^3)(1+z^4)$

$= (1-z+z^2-z^3)(1-z^3+z^4-z^7)$

$= (-z^4)(1-z^3+z^4+z^2) \quad (\because 1-z+z^2-z^3+z^4=0$ and $z^7 = z^5 \cdot z^2 = -z^2)$

$= (-z^4)(z) \quad (\because 1-z+z^2-z^3+z^4=0)$

$= -z^5 = 1$ as required.

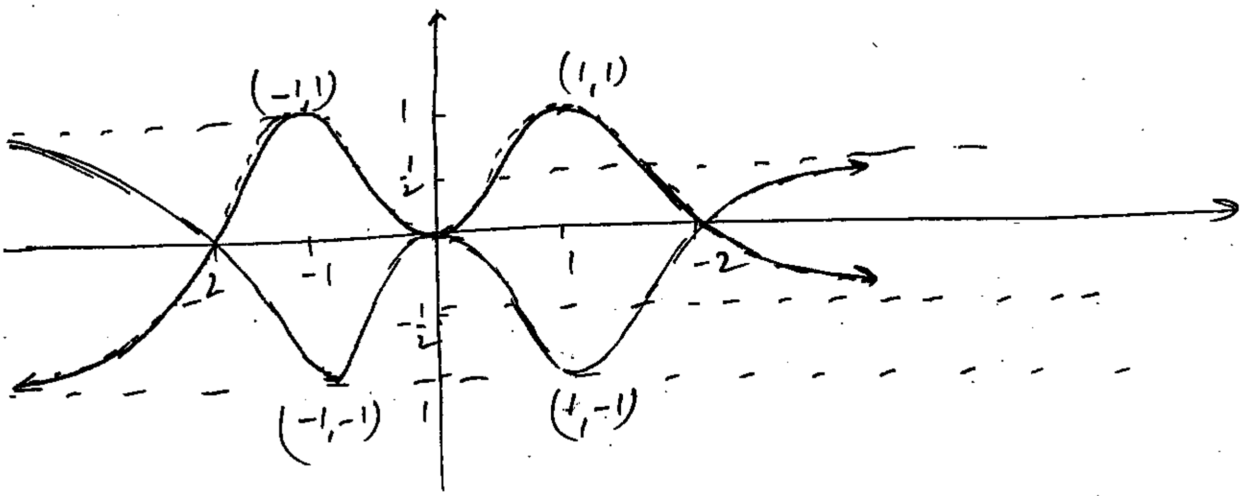
ANSWER SHEET

Name: ____________________

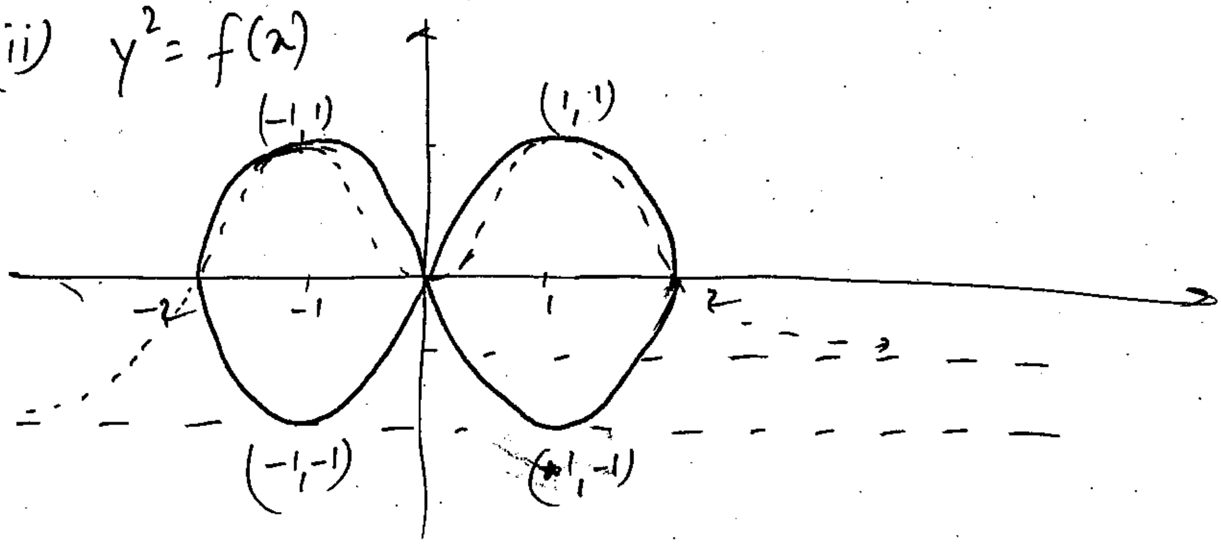
Teacher: ____________________

Question No. 3

(a) (i) $|y| = |f(x)|$



(ii) $y^2 = f(x)$



ANSWER SHEET

Name: ____________________

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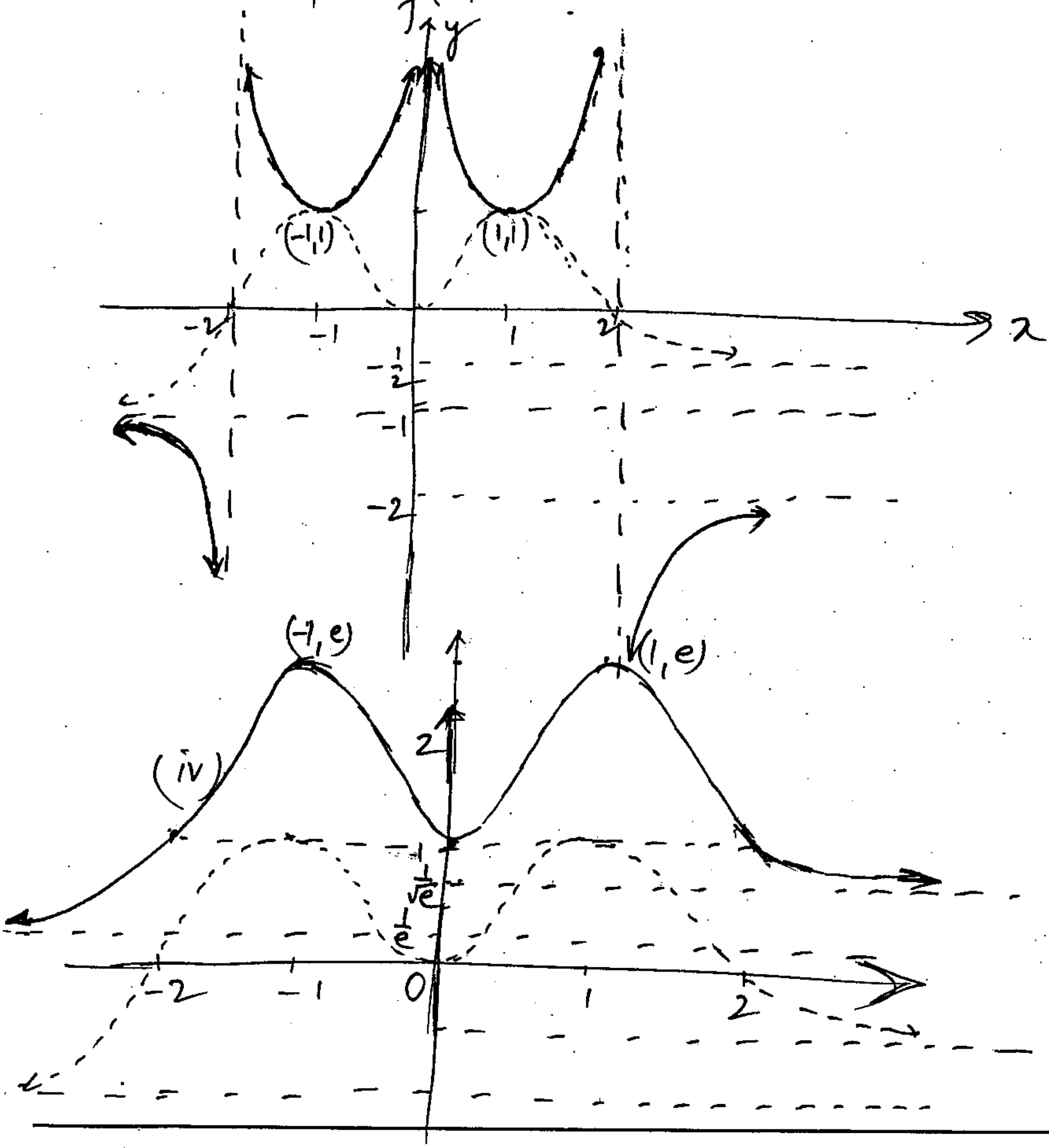
Question No. 3

(a) (iii) $y = \frac{1}{f(x)}$



ANSWER SHEET

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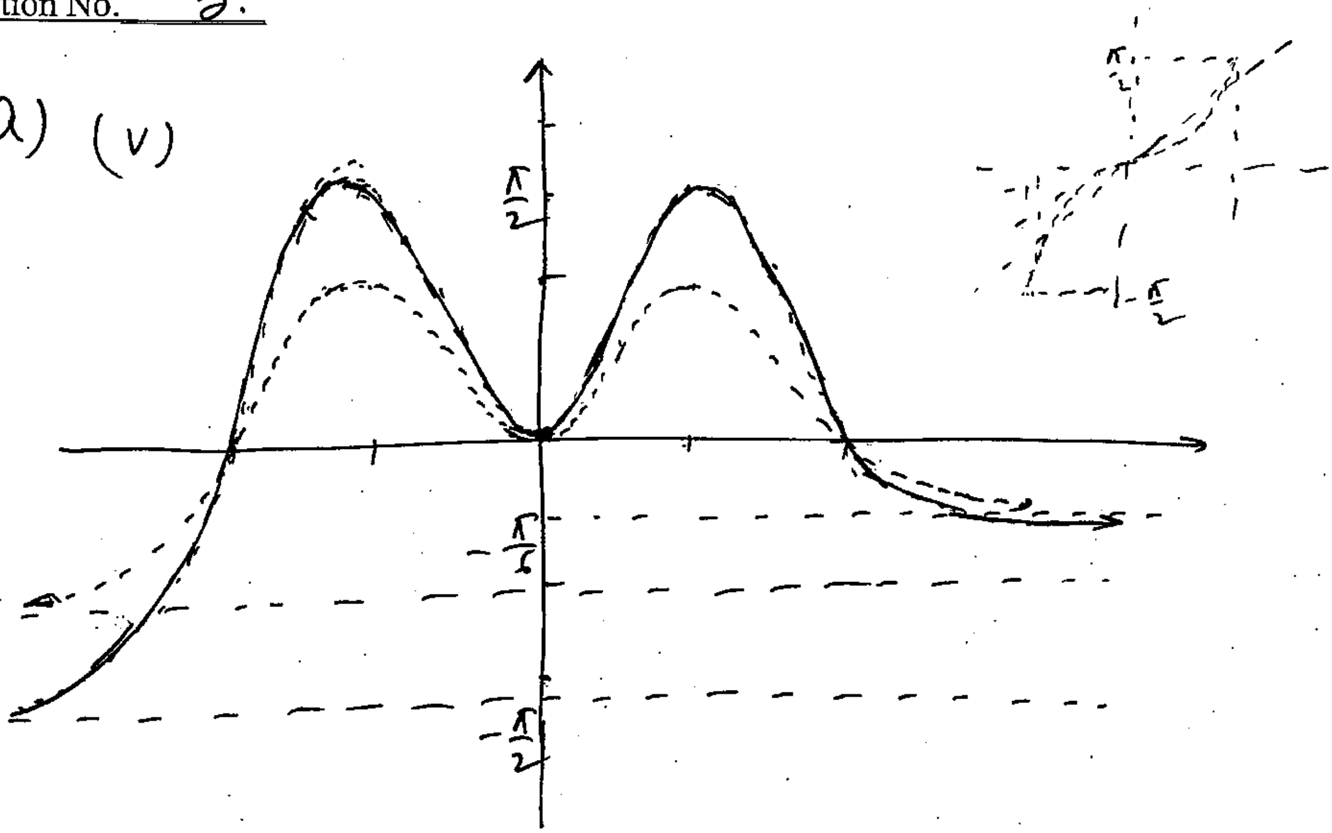
Question No. 3.

(a) (v)



ANSWER SHEET

Name: ______________________

Teacher: ______________________

Question No. 3 5

(b) (i) $I_n = \int_0^1 x^n \sqrt{1-x^2}\,dx$

$u = x^{n-1}$ $v' = x\sqrt{1-x^2}$

$u' = (n-1)x^{n-2}$ $v = -\frac{1}{2}(1-x^2)^{3/2} \cdot \frac{2}{3}$

$= -\frac{1}{3}(1-x^2)^{3/2}$

$$= \left[-\frac{x^{n-1}}{3}(1-x^2)^{3/2}\right]_0^1 + \int_0^1 \frac{(n-1)x^{n-2}}{3}(1-x^2)^{3/2}\,dx$$

$$= (0-0) + \frac{n-1}{3}\int_0^1 x^{n-2}(1-x^2)(1-x^2)^{1/2}\,dx$$

$$= \frac{n-1}{3}\int_0^1 \left(x^{n-2}\sqrt{1-x^2} - x^n\sqrt{1-x^2}\right)dx$$

$$= \frac{n-1}{3}\left(I_{n-2} - I_n\right)$$

$3I_n = (n-1)I_{n-2} - (n-1)I_n$

$3I_n + (n-1)I_n = (n-1)I_{n-2}$

$(n+2)I_n = (n-1)I_{n-2}$

$\therefore I_n = \frac{n-1}{n+2} I_{n-2}$

ANSWER SHEET

Name: ____________________

Teacher: ____________________

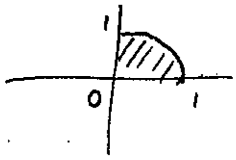
Question No. 3

(b) (iii) $I_4 = \int_0^1 x^4 \sqrt{1-x^2}\, dx$

$= \frac{3}{5} I_2$

$= \frac{3}{5} \left[\frac{1}{3} I_0 \right]$

$I_0 = \int_0^1 \sqrt{1-x^2}\, dx$



$= \frac{\pi}{4}(1)^2 = \frac{\pi}{4}$

$\therefore \int_0^1 x^4 \sqrt{1-x^2}\, dx = \frac{3}{5} \times \frac{1}{3} \times \frac{\pi}{4}$

$= \frac{\pi}{20}$

ANSWER SHEET

Name: ____________

Teacher: ____________

Question No. 4

(a) Assymptotes are $y = x$ and $y = -x$

(i) ∴ Rectangular hyperbola

$$x^2 - y^2 = a^2$$

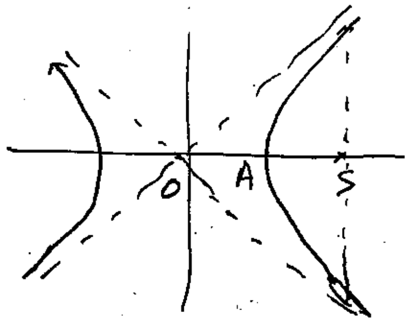
Sub $(5, 4)$

$$25 - 16 = a^2$$

$$a^2 = 9$$

$$\underline{a = 3}$$

Equation $\underline{x^2 - y^2 = 9}$



(ii) Eccentricity : $\sqrt{2}$

(iii) Length of principal axis : 6

(iv) Focus : $(\pm 3\sqrt{2}, 0)$

(v) Equation of directrix $x = \pm a/e$

$$x = \pm \frac{3}{\sqrt{2}}$$

(vi) Length of Latus Rectum: $2b^2/a = 2a^2/a$

$$= 2a = \underline{\underline{6}}$$

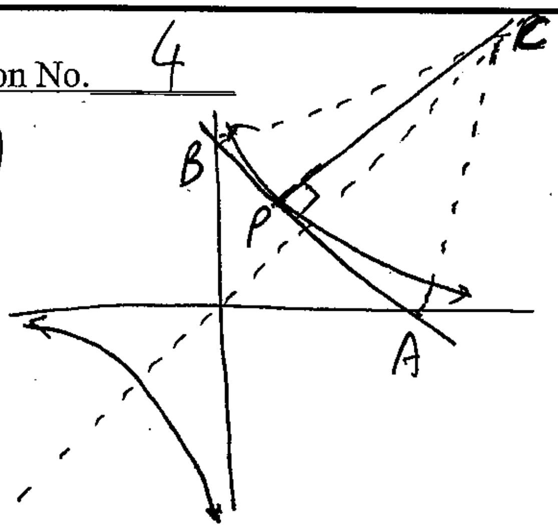
ANSWER SHEET

Name: ____________________

Teacher: ____________________

Question No. 4

(b)



(i) $y = c^2/x$

$\frac{dy}{dx} = -c^2/x^2$

at $x = cp$

$\frac{dy}{dx} = -\frac{c^2}{c^2p^2}$

$= -1/p^2$

Equation of tangent

$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$

$p^2y - cp = -x + cp$

or $\underline{x + p^2y = 2cp}$

(ii) $y = 0, \quad x = 2cp \quad \therefore A(2cp, 0)$

$x = 0, \quad y = 2cp/p^2$

$= 2c/p \quad \therefore B(0, 2c/p)$

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Question No. 4

(b) (iii) $M_N = p^2$

Equation of normal

$y - \frac{c}{p} = p^2(x - \cancel{2}cp)$

$py - c = p^3x - \cancel{2}cp^4$

or $\underline{p^3x - py = \cancel{2}cp^4 - c}$

(iv) $y = x$

$\therefore p^3x - px = c(p^4 - 1)$

$px(p^2-1) = c(p^2-1)(p^2+1)$

$\therefore \underline{x = \frac{c}{p}(p^2+1)} \Rightarrow (cp + \frac{c}{p})$

(v) $AC^2 = (cp + \frac{c}{p} - 2cp)^2 + (cp + \frac{c}{p} - \cancel{\frac{2c}{p}}^{0})^2$

$= (\frac{c}{p} - cp)^2 + (cp + \frac{c}{p})^2$

$= 2(\frac{c^2}{p^2} + c^2p^2)$

$BC^2 = (cp + \frac{c}{p} - 0)^2 + (cp + \frac{c}{p} - \frac{2c}{p})^2 = (cp + \frac{c}{p})^2 + (cp - \frac{c}{p})^2$

$= 2(c^2p^2 + c^2/p^2)$

$\therefore \underline{AC = BC}$

isosceles

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Question No. 5

(a) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1 \qquad P(a\cos\theta, b\sin\theta)$

(i) Differentiating w.r.t. x

$$\frac{2x}{a^2}+\frac{2y}{b^2}\cdot\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{b^2x}{a^2y}$$

When $x=a\cos\theta,\ y=b\sin\theta$

$$\frac{dy}{dx}=-\frac{b^2.a\cos\theta}{a^2.b\sin\theta}=-\frac{b\cos\theta}{a\sin\theta}$$

Equation of tangent

$$y-b\sin\theta=-\frac{b\cos\theta}{a\sin\theta}(x-a\cos\theta)$$

$$ay\sin\theta-ab\sin^2\theta=-bx\cos\theta+ab\cos^2\theta$$

$$bx\cos\theta+ay\sin\theta=ab(\cos^2\theta+\sin^2\theta)$$

$$bx\cos\theta+ay\sin\theta=ab$$

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Question No. 5

(a) (ii) $M_N = \frac{a\sin\theta}{b\cos\theta}$

Equation of normal

$y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$

$by\cos\theta - b^2\sin\theta\cos\theta = ax\sin\theta - a^2\sin\theta\cos\theta$

$ax\sin\theta - by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta$

(iii) For X:

$y = 0 \quad bx\cos\theta = ab$

$\boxed{x = \frac{a}{\cos\theta}}$

$x = 0,$

$ay\sin\theta = ab$

$y = \frac{b}{a\sin\theta}$

$X\left(0, \frac{b}{a\sin\theta}\right)$

For Y

$y = 0 \quad ax\sin\theta = (a^2 - b^2)\sin\theta\cos\theta$

$x = \frac{a^2 - b^2}{a}\cos\theta$

$-by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta$

$y = -\frac{a^2 - b^2}{b}\sin\theta$

$Y\left(0, -\frac{a^2 - b^2}{b}\sin\theta\right)$

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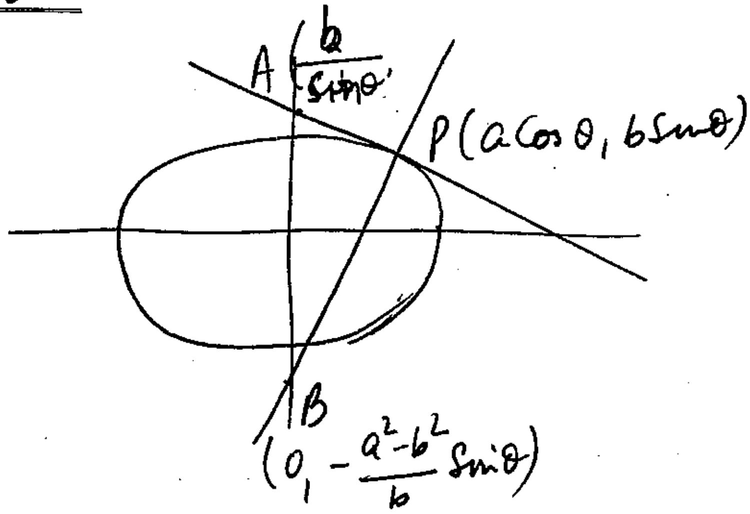
Question No. 5

(a) (iv)



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$b^2 = a^2(1-e^2)$$

$$a^2e^2 = a^2 - b^2$$

$$AB = \frac{b}{\sin\theta} + \frac{a^2-b^2}{b}\sin\theta$$

$$= \frac{b^2 + (a^2 - b^2)\sin^2\theta}{b\sin\theta}$$

$$= \frac{b^2 + a^2e^2\sin^2\theta}{b\sin\theta}$$

radius: $\frac{b^2 + a^2e^2\sin^2\theta}{2b\sin\theta}$

Midpt of AB

$x = 0$

$$y = \left(\frac{b}{\sin\theta} - \frac{a^2-b^2}{b}\sin\theta\right) \div 2$$

$$= \frac{b^2 - a^2e^2\sin^2\theta}{2b\sin\theta}$$

∴ Equation of circle:

$$x^2 + \left(y - \frac{b^2 - a^2e^2\sin^2\theta}{2b\sin\theta}\right)^2 = \left(\frac{b^2 + a^2e^2\sin^2\theta}{2b\sin\theta}\right)^2$$

Sub $x = \pm ae$, $y = 0$

LHS: $a^2e^2 + \left(\frac{b^2 - a^2e^2\sin^2\theta}{2b\sin\theta}\right)^2 = \frac{4a^2b^2e^2\sin^2\theta + (b^2 - a^2e^2\sin^2\theta)^2}{4b^2\sin^2\theta}$

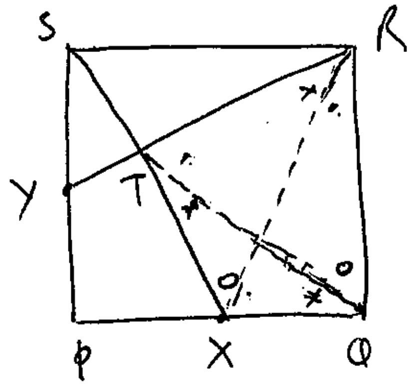
$$= \left(\frac{b^2 + a^2e^2\sin^2\theta}{2b\sin\theta}\right)^2$$

$= RHS$

∴ Circle passes through S & S'.

Question No. 5

(b)



(i) In Δ PXS and Δ YRS

PX = SY (half of equal sides of a square)

PS = SR (equal sides of square)

$\angle SPX = \angle SRY = 90^\circ$ ($\because$ PQRS is a square)

$\therefore \Delta PXS \equiv \Delta YRS$ (SAS)

$\therefore \angle PSX = \angle SRY$ (corresponding $\angle$'s of congruent Δ's)

In ΔSYT and ΔSRY

$\angle SYT = \angle SYR$ (common angle)

$\angle YST = \angle SRY$ (proven above)

$\therefore \angle STY = \angle YSR$ (angle sum of Δ)

$\therefore \angle STY = 90^\circ$ ($\because \angle YSR = 90^\circ$, internal angle of square)

$\angle STY = \angle RQX = 90^\circ$

$\therefore$ QRTX is a cyclic quadrilateral.

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Question No. 5

(b) (ii) Join XR

In ΔPSX and ΔQRX

$PX = QX$ (X is mid pt of PQ)

$PS = QR$ (equal sides of square)

$\angle SPX = \angle RQX = 90^\circ$

$\therefore \Delta SPX \equiv \Delta QRX$ (SAS)

$\therefore \angle PXS = \angle QXR$ (corresponding $\angle$'s)

$\angle PXS = \angle QRT$ (~~opp~~ exterior angle of cyclic quadrilateral equal to opp. interior $\angle$)

$\angle QXR = \angle QTR$ (angles in same segment)

$\therefore \angle QRT = \angle QTR$

$\therefore QR = QT$ (equal sides opposite to equal angles of a ΔQRT)

ANSWER SHEET

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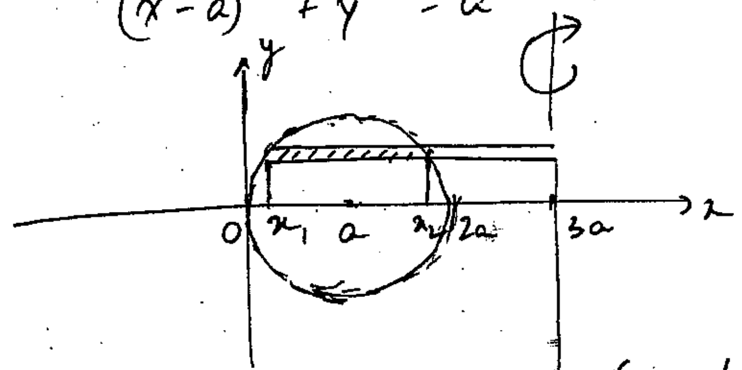
Question No. 6

(a) $x^2-2ax+y^2=0$

or $x^2-2ax+a^2+y^2=a^2$

$(x-a)^2+y^2=a^2$



$(x-a)^2 = a^2-y^2$

$x-a = \pm\sqrt{a^2-y^2}$

$x = a\pm\sqrt{a^2-y^2}$

$x_2+x_1 = 2a$

$x_2-x_1 = 2\sqrt{a^2-y^2}$

Let a section of thickness δy be taken $\perp$ to $x=3a$.
Volume of the section when rotated about $x=3a$ is

$\delta V = \pi\left[(3a-x_1)^2-(3a-x_2)^2\right]\delta y$

$= \pi\left[(6a-x_1-x_2)(-x_1+x_2)\right]\delta y$

$= \pi(6a-2a)(2\sqrt{a^2-y^2})\,\delta y$

$= 8a\pi\sqrt{a^2-y^2}\,\delta y$

$\therefore V = \int_{-a}^{a} 8a\pi\sqrt{a^2-y^2}\,dy$

$= 8a\pi\int_{-a}^{a}\sqrt{a^2-y^2}\,dy$

$= 8a\pi\cdot\frac{\pi a^2}{2} = \underline{\underline{4a^3\pi^2}}\ u^3.$

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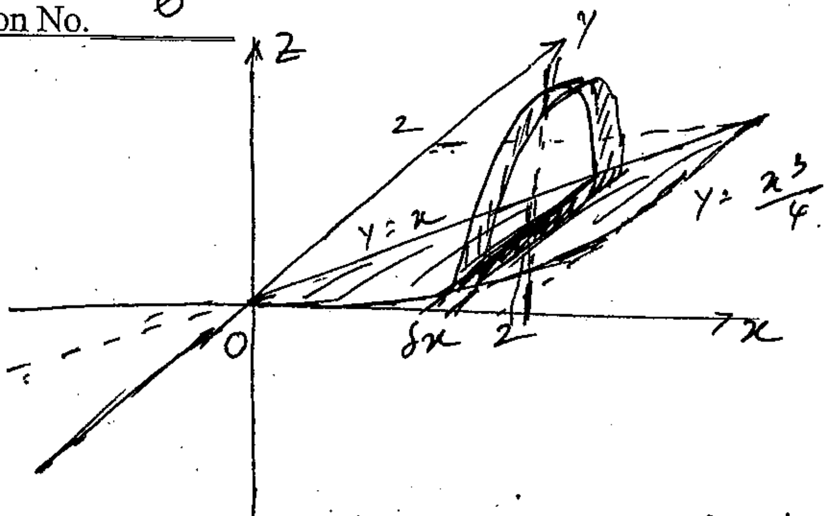
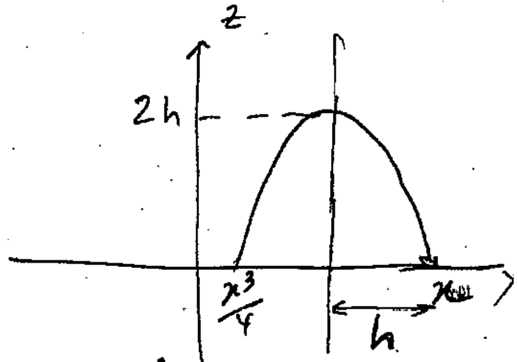
Question No. 6

(b)





Area of section

Let a section of thickness δx be taken $\perp$ to x-axis

Using Simpson's rule, the area of the section is

$$A(x) = \frac{h}{3}\left(0 + 4(2h) + 0\right)$$

where $h = \dfrac{x - x^3/4}{2} = \dfrac{4x - x^3}{8}$

$$\therefore A(x) = \frac{4x - x^3}{8 \times 3}\left(8 \cdot \frac{4x - x^3}{8}\right)$$

$$= \frac{(4x - x^3)^2}{24}$$

$$\delta V = A(x)\,\delta x$$

$$\therefore V = \int_0^2 \frac{(4x - x^3)^2}{24}\,dx$$

$$= \frac{1}{24}\int_0^2 \left(16x^2 - 8x^4 + x^6\right)dx$$

$$= \frac{1}{24}\left[\frac{16x^3}{3} - \frac{8x^5}{5} + \frac{x^7}{7}\right]_0^2 = \frac{1}{24}\left[\frac{16}{3}(8) - \frac{8}{5}(32) + \frac{1}{7}(128) - 0\right]$$

$$= \frac{128}{315}\,u^3.$$

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Question No. 6

(c) $u_n = u_{n-1} + 6u_{n-2}$, $n \geq 3$, $u_1 = 1$, $u_2 = -12$

To prove
$$u_n = -6\left[(-2)^{n-2} + 3^{n-2}\right]$$

Step 1 : prove true for $n=1$ and $n=2$

$u_1 = -6\left((-2)^{-1} + 3^{-1}\right)$
$= -6\left[-\frac{1}{2} + \frac{1}{3}\right]$
$= -6 \cdot \frac{-1}{6}$
$= 1$ True

$u_2 = -6\left[(-2)^0 + 3^0\right]$
$= -6(1+1)$
$= -12$ True.

Hence true for $n=1$ and $n=2$

Step 2 Assume true for $n=k$ & $n=k+1$, $k \geq 1$

ie $u_k = -6\left[(-2)^{k-2} + (3)^{k-2}\right]$, $u_{k+1} = -6\left[(-2)^{k-1} + 3^{k-1}\right]$

Step 3 : prove true for $n = k+2$

ie. $u_{k+2} = -6\left[(-2)^k + 3^k\right]$

LHS $= u_{k+2} = u_{k+1} + 6u_k$
$= -6\left[(-2)^{k-1} + 3^{k-1}\right] - 36\left[(-2)^{k-2} + 3^{k-2}\right]$ from step 2
$= -6\left[\frac{(-2)^k}{-2} + \frac{3^k}{3}\right] - 36\left[\frac{(-2)^k}{4} + \frac{3^k}{9}\right]$
$= 3(-2)^k + 2(3^k) - 9(-2)^k - 4(3^k)$
$= -6(-2)^k - 6(3^k)$
$= -6\left[(-2)^k + 3^k\right] = RHS.$

Hence true for $n = k+2$

Step 4 : By the second principle of mathematical induction; it is true for all $n \geq 1$

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Question No. 7

(a) (i) $\cos 4\theta + i\sin 4\theta = (\cos\theta + i\sin\theta)^4$

$$= \cos^4\theta + 4i\cos^3\theta\sin\theta + 6\cos^2\theta(i\sin\theta)^2 + 4\cos\theta(i\sin\theta)^3 + (i\sin\theta)^4$$

$$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$$

Equating real and imaginary parts

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \quad - (1)$$

$$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta \quad - (2)$$

(2) ÷ (1)

$$\tan 4\theta = \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$$

Dividing each term on RHS by $\cos^4\theta$

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

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Question No. 7

(a) (ii) $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

$$x^4 - 6x^2 + 1 = 4x - 4x^3$$

or $$\frac{4x - 4x^3}{1 - 6x^2 + x^4} = 1$$

Let $x = \tan\theta$

then $$\frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta} = 1$$

Using the identity from part (i)

$$\tan 4\theta = 1$$

$$4\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$$

$$\therefore \theta = \frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}$$

$$= \frac{\pi}{16}, \frac{5\pi}{16}, -\frac{7\pi}{16}, -\frac{3\pi}{16}$$

$$\therefore x = \tan\frac{\pi}{16}, \tan\frac{5\pi}{16}, \tan\left(-\frac{7\pi}{16}\right), \tan\left(-\frac{3\pi}{16}\right)$$

$$= \tan\frac{\pi}{16}, \tan\frac{5\pi}{16}, -\tan\frac{7\pi}{16}, -\tan\frac{3\pi}{16}.$$

Product of roots = 1 from the polynomial equation

$$\therefore \tan\frac{\pi}{16} \times \tan\frac{5\pi}{16} \times -\tan\frac{7\pi}{16} \times -\tan\frac{3\pi}{16} = 1$$

or $$\tan\frac{\pi}{16} \times \tan\frac{5\pi}{16} \times \tan\frac{7\pi}{16} \times \tan\frac{3\pi}{16} = 1$$

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Question No. 7

(b) $P(x) = x^4 + x^3 - 16x^2 - 4x + 48 = 0$

let the roots be $\alpha, \beta, \gamma, \delta$, let $\alpha\beta = 6$

$\alpha + \beta + \gamma + \delta = -1 \quad - (1)$

$\alpha\beta + \beta\gamma + \gamma\delta + \alpha\delta + \alpha\gamma + \beta\delta = -16 \quad - (2)$

$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \alpha\beta\delta = 4 \quad - (3)$

$\alpha\beta\gamma\delta = 48 \quad - (4)$

$\gamma\delta = 48/6 = 8$

$P(x) = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta) = 0$

or $(x^2 - (\alpha+\beta)x + \alpha\beta)(x^2 - (\gamma+\delta)x + \gamma\delta) = 0$.

from (3) $6\gamma + 8\beta + 8\alpha + 6\delta = 4$

or $8(\alpha+\beta) + 6(\gamma+\delta) = 4 \quad - (5)$

(1) $\times 8$ $\quad 8(\alpha+\beta) + 8(\gamma+\delta) = -8 \quad - (6)$

(5) $-$ (6)

$-2(\gamma+\delta) = 12$

$\gamma + \delta = -6$

$\alpha + \beta = 5$

$\therefore P(x) = (x^2 - 5x + 6)(x^2 + 6x + 8) = 0$

or $(x-3)(x-2)(x+4)(x+2) = 0$

$\therefore x = 3, 2, -4, -2$

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Question No. 7

(c) (i) $3x^4 - 4x^3 + m = 0$ ~~⇒~~

let $f(x) = 3x^4 - 4x^3 + m$

$f'(x) = 12x^3 - 12x^2$

$f''(x) = 36x^2 - 24x$

$f'(x) = 0 \quad 12x^2(x-1) = 0$ when $x = 1$

$f''(x) > 0 \quad \therefore$ min. pt.

$x = 0$ or $x = 1$

$f(0) = m, \ f(1) = (3 - 4 + m) = (m-1)$

x	0^-	0	0^+
$f'(x)$	$-$	0	$-$

say $(\frac{1}{2})$

$\therefore x = 0$ is a horizontal poi

$f''(x) \quad + \quad 0 \quad -$

For $f(x) = 0$ to have no real roots,

the poi and st. point must lie on the same side of x-axis ie above the x-axis ($\because y$-int. > 0; (ie $m > 0$)

ie $f(0) \times f(1) > 0$

$(m) \times (m-1) > 0$

$\therefore \underline{m < 0 \text{ or } m > 1}$

$\therefore m > 1$ for no real roots

($\because y$-int below x-axis $\therefore$ there will be two intercepts.

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Question No. 7

(c) (ii) when $m=1$

$f(0)=1$, $f(1)=0$.

∴ st. point lies on the x-axis , poi is above the x-axis

There is one real root.

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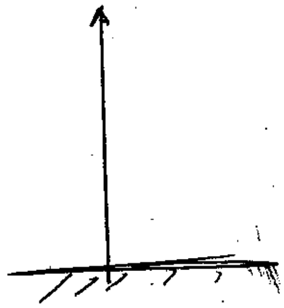
Question No. 8

(a) (i)



$$\Sigma F = m\ddot{x} = -mg - mkv^2$$

$$\ddot{x} = v\frac{dv}{dx} = -g - kv^2$$

$$dx = -\frac{v\,dv}{g+kv^2}$$

$$x = \int -\frac{v\,dv}{g+kv^2}$$

$$= -\frac{1}{2k}\ln(g+kv^2) + C$$

when $x=0$, $v=u$

$$0 = -\frac{1}{2k}\ln(g+ku^2) + C$$

$$C = \frac{1}{2k}\ln(g+ku^2)$$

$$\therefore\ x = -\frac{1}{2k}\ln(g+kv^2) + \frac{1}{2k}\ln(g+ku^2)$$

$$= \frac{1}{2k}\ln\frac{g+ku^2}{g+kv^2}$$

for max ht H, $v=0$

$$H = \frac{1}{2k}\ln\frac{g+ku^2}{g}$$

$$\text{or } \underline{\underline{H = \frac{1}{2k}\ln\left(1+\frac{ku^2}{g}\right)}}$$

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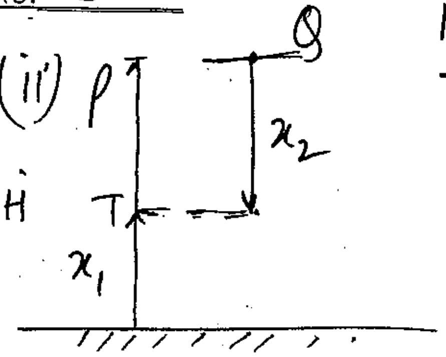
Question No. 8

(a) (ii)



For Q: $\Sigma F = m\ddot{x} = mg - mkv^2$

for terminal velocity $\ddot{x} = 0$

$g = kV^2$

or $V = \sqrt{g/k}$

$$v\frac{dv}{dx} = g - kv^2$$

$$dx = \frac{v\,dv}{g-kv^2}$$

$$x = \int \frac{v\,dv}{g-kv^2}$$

$$= -\frac{1}{2k}\ln(g-kv^2) + C$$

when $x = 0$, $v = 0$

$$0 = -\frac{1}{2k}\ln g + C \quad \therefore C = \frac{1}{2k}\ln g$$

$$\therefore x = -\frac{1}{2k}\ln(g-kv^2) + \frac{1}{2k}\ln g$$

$$= \frac{1}{2k}\ln\frac{g}{g-kv^2}$$

Let the particles collide at T when for P, $x = x_1$, $v = v_1$, and for Q, $x = x_2$, $v = v_2$ $\Rightarrow$ $x_1 + x_2 = H$.

from (i)a $x_1 = \frac{1}{2k}\ln\frac{g+ku^2}{g+kv_1^2}$ $\quad x_2 = \frac{1}{2k}\ln\frac{g}{g-kv_2^2}$

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Question No. 8

a)(ii) $x_1 + x_2 = H = \frac{1}{2k} \ln\left(1 + \frac{ku^2}{g}\right)$

$$\frac{1}{\cancel{2k}} \ln\left(\frac{g+ku^2}{g}\right) = \frac{1}{\cancel{2k}} \ln\left(\frac{g+ku^2}{g+kv_1^2}\right) + \frac{1}{\cancel{2k}} \ln\left(\frac{g}{g-kv_2^2}\right)$$

$$\frac{g\cancel{+ku^2}}{g} = \frac{\cancel{g+ku^2}}{g+kv_1^2} \cdot \frac{g}{g-kv_2^2}$$

$$(g+kv_1^2)(g-kv_2^2) = g^2$$

$$k^2\left(\frac{g}{k}+v_1^2\right)\left(\frac{g}{k}-v_2^2\right) = g^2$$

$$\left(\frac{g}{k}+v_1^2\right)\left(\frac{g}{k}-v_2^2\right) = \frac{g^2}{k^2}$$

or $(V^2+v_1^2)(V^2-v_2^2) = V^4$

$$V^4 + V^2(v_1^2 - v_2^2) - v_1^2 v_2^2 = V^4$$

or $V^2(v_1^2 - v_2^2) = v_1^2 v_2^2$

Dividing by $V^2 v_1^2 v_2^2$

$$\frac{1}{v_2^2} - \frac{1}{v_1^2} = \frac{1}{V^2}$$

ANSWER SHEET

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Question No. 8

(b) $x^2 + y^2 = xy + 3$ — ①

Differentiating implicitly with respect to x

$2x + 2y\frac{dy}{dx} = x\frac{dy}{dx} + y$

$(x - 2y)\frac{dy}{dx} = 2x - y$

$\therefore \frac{dy}{dx} = \frac{2x-y}{x-2y}$

for stationary points $y' = 0$

$2x - y = 0 \quad \therefore y = 2x$ — ②

Solving ① & ② simultaneously, sub $y = 2x$ into ①

$x^2 + 4x^2 = 2x^2 + 3$

$3x^2 = 3 \quad x = 1, \; y = 2$

or $x = -1, \; y = -2$

$y'' = \frac{(x-2y)(2-y') - (2x-y)(1-2y')}{(x-2y)^2}$

(i) When $y' = 0$, $x = 1$, $y = 2$

$y'' = \frac{(1-4)(2)}{(1-4)^3} < 0$

$\therefore (1,2)$ is max pt

(ii) When $y' = 0$, $x = -1$, $y = -2$

$y'' = \frac{(-1+4)(2)}{(-1+4)^3} > 0$

$\therefore (-1,-2)$ is min pt

ANSWER SHEET

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Teacher: ____________________

Question No. 8

(C) (i) For the branch $X - P - Q - Y$

current will not flow if P defective or Q defective or P and Q defective.

$\therefore$ P(no current flow through $X-P-Q-Y$)

$$= p(1-p) + (1-p)p + p \cdot p$$
$$= p - p^2 + p - p^2 + p^2$$
$$= 2p - p^2 = p(2-p)$$

Similarly P(no current flow through $X-R-S-Y$)

$$= 2p - p^2 = p(2-p)$$

$\therefore$ P(no current flow from X to Y) $= \underline{(2p-p^2)^2} = p^2(2-p)^2$

(ii) let q = probability of no current through any one of the four circuits. then $q = (2p-p^2)^2 = p^2(2-p)^2$

then from above.

P(no current from A to B) $= \cancel{(2q-q^2)}^2$ where $\cancel{q = (2p-p^2)^2}$

$$= \cancel{\left[2(2p-p^2)^2 - (2p-p^2)^4\right]}^2$$

Probability : $p^4(2-p)^4\left[2 - p^2(2-p)^2\right]^2 = p^4(2-p)^4\left[2 - p^2(4-4p+p^2)\right]^2$

$$= \underline{p^4(2-p)^4(2-4p^2+4p^3-p^4)^2}$$